Existence and Uniqueness Theorem for Systems of First Order Linear IVP

Theorem 4.1

Consider the initial value problem

$$\mathbf{y}'(t) = P(t)\mathbf{y}(t) + \mathbf{g}(t), \qquad \mathbf{y}(t_0) = \mathbf{y}_0,$$

where $\mathbf{y}(t)$, $P(t)$, $\mathbf{g}(t)$, and $\mathbf{y}_0$ are defined as in equation (2). Let the n^2 components of $P(t)$ and the n components of $\mathbf{g}(t)$ be continuous on the interval (a, b), and let t_0 be in (a, b). Then the initial value problem has a unique solution that exists on the entire interval (a, b).

(Theorem 4.1 page 224 textbook)

Example:

Consider the initial value problem

$$y_1' = (\sin 2t)y_1 + \frac{t}{t^2 - 2t - 8}y_2 + 4, \qquad y_1(1) = 2$$

$$y_2' = (\ln|t+1|)y_1 + e^{-2t}y_2 + \sec t, \qquad y_2(1) = 0.$$

Determine the largest t-interval on which Theorem 4.1 guarantees the existence of a unique solution of this problem.

$$\vec{y}(1) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\vec{y}(t) = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Rightarrow \vec{y}\,'(t) = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \underbrace{\begin{bmatrix} \sin(2t) & \frac{t}{(t-4)(t+2)} \\ \ln|t+1| & e^{-2t} \end{bmatrix}}_{P(t)} \underbrace{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}}_{y(t)} + \underbrace{\begin{bmatrix} 4 \\ \sec(t) \end{bmatrix}}_{\vec{G}(t)} \qquad \vec{y}(1) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

(3) Look for possible discontinuities of $P(t)$ and components of $\vec{G}(t)$

$t \neq 4, -2, -1, \frac{\pi}{2}, -\frac{\pi}{2}$

(4) Place $t_0 = 1$ in $\mathbb{R}$

$(-1, \; t_0 = 1, \; \frac{\pi}{2})$

(5) Conclusion: theorem guarantees a unique solution over $(-1, \frac{\pi}{2})$

Existence and Uniqueness Theorem for second order linear IVP

Let $p(t)$, $q(t)$, and $g(t)$ be continuous functions on the interval (a,b), and let t_0 be in (a,b). Then the initial value problem

$$y'' + p(t)y' + q(t)y = g(t), \qquad y(t_0) = y_0, \qquad y'(t_0) = y_0'$$

has a unique solution defined on the entire interval (a,b).

(Theorem 3.1 page 111 textbook)

Ex: Determined the largest t-interval in which we can guarantee the existence and uniqueness of a solution of the IVP

- $ty'' + cos(t)y + t^2y = t, \quad y(-1) = 1,\ y'(-1) = 2$

$$\vec{y}(t) = \begin{pmatrix} y \\ y' \end{pmatrix} \in \mathbb{R}^2 \Rightarrow \vec{y}\,' = \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_2 \\ 1 + \underbrace{\left(-t - \frac{\cos(t)}{t}\right)}\, y_1 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -t - \frac{\cos(t)}{t} & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

get from scalar D.E.

① SF

② $y'' = 1 - ty - \frac{\cos(t)}{t} y = 1 - ty_1 - \frac{\cos(t)}{t} y_1$

$y_1 = y$ from def, ie from of vector $\vec{y}$

Theorem 4.1 $t \neq 0$ IVP 1)

Look at $t_0 = -1$ IVP 2)

$t_0 = 2$

~~(-1 0)~~

Theorem 4.1 garantees the existence of a unique solution on $(-\infty, 0)$ and $(0, \infty)$

(IVP 1) (IVP 2)

- $ty'' + cos(t)y + t^2y = t, \quad y(1) = 1,\ y'(1) = 2$

$$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y \\ y' \end{pmatrix} \Rightarrow \vec{y}\,' = \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} y_2 \\ 1 + \left(\frac{-\cos(t)}{t} - t\right) y_1 \end{pmatrix} = \vec{F}(t, \vec{y}) = \begin{pmatrix} f_1(t, \vec{y}) \\ f_2(t, \vec{y}) \end{pmatrix}$$

from the scalar D.E.

$y'' = \frac{1 - \cos(t)}{t} y - ty = 1 + \left(\frac{-\cos(t)}{t} + t\right) y_1$

$y = y_1$

$\vec{y}(t_0) = y_0 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$y_0 = \begin{pmatrix} y_1(t_0) \\ y_2(t_0) \end{pmatrix}$

unique solution to IVP on $(0, +\infty)$

① Can I write $\vec{F}(t\vec{y})$ as $P(t) + \vec{G}(t)$?

$$\vec{y}\,' = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{\cos(t)}{t} - t & 0 \end{bmatrix}}_{P(t)} \underbrace{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}}_{y} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\vec{G}(t)} \Rightarrow \underbrace{\vec{y}\,' = p(t)\vec{y} + \vec{G}(t)}_{\text{linear system}}$$

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Ex: Can you apply the same result to the IVP $ty'' + cos(y)y + t^2y = t, \quad y(-1) = 1,\ y'(-1) = 2$? Explain your answer.

$$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y \\ y' \end{pmatrix} \Rightarrow \vec{y}\,' = \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} y_2 \\ y'' \end{pmatrix} = \begin{pmatrix} y_2 \\ 1 - ty_1 - \frac{\cos(y_1)\, y_1}{t} \end{pmatrix}$$

$\vec{y}\,' = \vec{F}(t\vec{y})$

↳ can't find a $P(t)$ and $\vec{G}(t)$ as the system is non-linear.

No